

Nonlinear Analysis: Modelling and Control, 2001, v. 6, No. 1, 3-8

Asymptotic of the Joint Distribution of Multivariate Extrema

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Received: 16.01.2001

Accepted: 14.03.2001

Abstract

Let W_n and Z_n be a bivariate extrema of independent identically distributed bivariate random variables with a distribution function F . In this paper the nonuniform estimate of convergence rate of the joint distribution of the normalized and centralized minima and maxima is obtained.

Keywords: multivariate extreme value, limit distributions, convergence rate

1 Introduction

Let $\{X_j = (X_{1,j}, X_{2,j}), j \geq 1\}$ be independent and identically distributed bivariate random variables with a distribution function

$$F(x_1, x_2) = P(X_{1,j} < x_1, X_{2,j} < x_2) \quad \forall j \geq 1,$$

and let $x = (x_1, x_2)$ be an arbitrary point of the Euclidean space. We define by

$$x + y = (x_1 + y_1, x_2 + y_2),$$

$$xy = (x_1 y_1, x_2 y_2),$$

$$\frac{x}{y} = \left(\frac{x_1}{y_1}, \frac{x_2}{y_2} \right)$$

the arithmetical operations on vectors. By the inequality $x < y$ we mean the system of inequalities $x_i < y_i$ ($i = 1, 2$).

We form bivariate maxima and minima

$$Z_n = (\max(X_{1,1}, \dots, X_{1,n}), \max(X_{2,1}, \dots, X_{2,n})),$$

$$W_n = (\min(X_{1,1}, \dots, X_{1,n}), \min(X_{2,1}, \dots, X_{2,n})).$$

Let

$$\{a_n = (a_{1,n}, a_{2,n}), n \geq 1\}, \quad \{b_n = (b_{1,n}, b_{2,n}) > (0,0), n \geq 1\}$$

be sequences of centralizing and normalizing vectors. If the distribution function F and sequences of centralizing and normalizing vectors are such that the following limit exists:

$$\lim_{n \rightarrow \infty} n(1 - F(a_n + b_n y)) = u(y_1, y_2) > 0,$$

then

$$\lim_{n \rightarrow \infty} P(Z_n < a_n + b_n y) = H(y) = e^{-u(y_1, y_2)}, \quad (1)$$

where $H(y)$ is a nondegenerate distribution function of two variables.

Let

$$\{c_n = (c_{1,n}, c_{2,n}), n \geq 1\}, \quad \{d_n = (d_{1,n}, d_{2,n}) > (0,0), n \geq 1\}$$

be sequences of centralizing and normalizing vectors. If the distribution function F and sequences of centralizing and normalizing vectors are such that the following limits exist

$$\lim_{n \rightarrow \infty} nF_1(c_{1,n} + d_{1,n}x_1) = z_1(x_1) > 0,$$

$$\lim_{n \rightarrow \infty} nF_2(c_{2,n} + d_{2,n}x_2) = z_2(x_2) > 0,$$

$$\lim_{n \rightarrow \infty} n(F_1(c_{1,n} + d_{1,n}x_1) + F_2(c_{2,n} + d_{2,n}x_2) - F(c_n + d_n x)) = z(x_1, x_2) > 0,$$

then

$$\lim_{n \rightarrow \infty} P(W_n < c_n + d_n x) = L(x) = 1 - e^{-z_1(x)} - e^{-z_2(x_2)} + e^{-z(x_1, x_2)}, \quad (2)$$

where $L(x)$ is a nondegenerate distribution function of two variables, and $F_1(x_1)$, $F_2(x_2)$ are marginal distribution functions of the distribution function $F(x)$.

The necessary and sufficient conditions for the convergence of normalized maxima and normalized minima are formulated in [3].

Suppose that conditions (1) and (2) hold true. Since the bivariate maxima and bivariate minima are asymptotically independent (see [2]), then

$$\lim_{n \rightarrow \infty} P(Z_n < a_n + b_n y, W_n < c_n + d_n x) = H(y)L(x). \quad (3)$$

In this paper we are going to obtain a nonuniform estimate of the convergence rate in (3). It will generalize the result obtained in [1].

2 Main result

Let us denote

$$\begin{aligned}
\Delta_{[c_n+d_n x, a_n+b_n y]} F &= F(a_n + b_n y) + F(c_n + d_n x) - \\
&- F(c_{1,n} + d_{1,n} x_1, a_{2,n} + b_{2,n} y_2) - F(a_{1,n} + b_{1,n} y_1, c_{2,n} + d_{2,n} x_2), \\
u_{1,n}(y) &= n(1 - F(a_n + b_n y)), \\
v_{1,n}(y) &= u_{1,n}(y) - u(y), \\
u_{2,n}(y, x_1) &= n(1 - F(a_n + b_n y) + F(c_{1,n} + d_{1,n} x_1, a_{2,n} + b_{2,n} y_2)), \\
v_{2,n}(y, x_1) &= u_{2,n}(y, x_1) - u(y) - z_1(x_1), \\
u_{3,n}(y, x_2) &= n(1 - F(a_n + b_n y) + F(a_{1,n} + b_{1,n} y_1, c_{2,n} + d_{2,n} x_2)), \\
v_{3,n}(y, x_2) &= u_{3,n}(y, x_2) - u(y) - z_2(x_2), \\
u_{4,n}(y, x) &= n(1 - \Delta_{[c_n+d_n x, a_n+b_n y]} F) \\
v_{4,n}(y, x) &= u_{4,n}(y, x) - u(y) - z(x).
\end{aligned}$$

Theorem. Suppose that (3) holds, and

$$a_n + b_n y > c_n + d_n x.$$

For all x, y , for which

$$\begin{aligned}
|v_{1,n}(y)| &\leq \log 2, & |v_{2,n}(y, x_1)| &\leq \log 2, \\
|v_{3,n}(y, x_2)| &\leq \log 2, & |v_{4,n}(y, x)| &\leq \log 2,
\end{aligned}$$

the following estimate holds true:

$$\begin{aligned}
|P(Z_n < a_n + b_n y, W_n < c_n + d_n x) - H(y)L(x)| &\leq \\
&\leq \Delta_{1,n}(y) + \Delta_{2,n}(y, x_1) + \Delta_{3,n}(y, x_2) + \Delta_{4,n}(y, x).
\end{aligned}$$

Here

$$\begin{aligned}
\Delta_{1,n}(y) &= \frac{u_{1,n}^2(y) e^{-u_{1,n}(y)}}{2(n-1)} + e^{-u(y)} (|v_{1,n}(y)| + v_{1,n}^2(y)), \\
\Delta_{2,n}(y, x_1) &= \frac{u_{2,n}^2(y, x_1) e^{-u_{2,n}(y, x_1)}}{2(n-1)} + e^{-u(y)-z_1(x_1)} (|v_{2,n}(y, x_1)| + v_{2,n}^2(y, x_1)), \\
\Delta_{3,n}(y, x_2) &= \frac{u_{3,n}^2(y, x_2) e^{-u_{3,n}(y, x_2)}}{2(n-1)} + e^{-u(y)-z_2(x_2)} (|v_{3,n}(y, x_2)| + v_{3,n}^2(y, x_2)), \\
\Delta_{4,n}(y, x) &= \frac{u_{4,n}^2(y, x) e^{-u_{4,n}(y, x)}}{2(n-1)} + e^{-u(y)-z(x)} (|v_{4,n}(y, x)| + v_{4,n}^2(y, x)).
\end{aligned}$$

Proof. We have

$$\begin{aligned} P(Z_n < y, W_n < x) &= P(Z_n < y) - P(Z_n < y, W_{1,n} \geq x_1) - \\ &- P(Z_n < y, W_{2,n} \geq x_2) + P(Z_n < y, W_n \geq x) = \\ &= F^n(y) - (F(y) - F(x_1, y_2))^n - (F(y) - F(y_1, x_2))^n + (\Delta_{[x,y]} F)^n, \end{aligned}$$

if $y > x$, and

$$P(Z_n < y, W_n < x) = P(Z_n < y) = F^n(y),$$

if $y \leq x$. Here $W_{1,n}, W_{2,n}$ are components of the bivariate minima W_n .

Let $a_n + b_n y > c_n + d_n x$. We have

$$\begin{aligned} |P(Z_n < a_n + b_n y, W_n < c_n + d_n x) - H(y)L(x)| &\leq \\ &\leq |F^n(a_n + b_n y) - e^{-u(y)}| + \\ &+ |(F(a_n + b_n y) - F(c_{1,n} + d_{1,n}x_1, a_{2,n} + b_{2,n}y_2))^n - e^{-u(y) - z_1(x_1)}| + \\ &+ |(F(a_n + b_n y) - F(a_{1,n} + b_{1,n}y_1, c_{2,n} + d_{2,n}x_2))^n - e^{-u(y) - z_2(x_2)}| + \\ &+ |(\Delta_{[c_n + d_n x, a_n + b_n y]} F)^n - e^{-u(y) - z(x)}|. \end{aligned} \quad (4)$$

Let us estimate the first summand of the right – hand side of the inequality

(4). We have

$$\begin{aligned} |F^n(a_n + b_n y) - e^{-u(y)}| &\leq \\ &\leq |F^n(a_n + b_n y) - e^{-u_{1,n}(y)}| + |e^{-u_{1,n}(y)} - e^{-u(y)}|. \end{aligned} \quad (5)$$

Let us estimate the second summand of the right – hand side of the inequality (5). Applying the inequality

$$0 \leq e^{-x} - \left(1 - \frac{x}{n}\right)^n \leq \frac{x^2 e^{-x}}{2(n-1)} \quad (0 \leq x \leq n),$$

we get

$$\begin{aligned} |F^n(a_n + b_n y) - e^{-u_{1,n}(y)}| &= \\ &= \left| \left(1 - \frac{u_{1,n}(y)}{n}\right)^n - e^{-u_{1,n}(y)} \right| \leq \\ &\leq \frac{u_{1,n}^2(y) e^{-u_{1,n}(y)}}{2(n-1)}. \end{aligned} \quad (6)$$

Let us estimate the first summand of the right – hand side of the inequality

(5). Applying the inequality

$$|e^{-y} - e^{-x}| \leq e^{-x} (|x - y| + (x - y)^2) \quad (|x - y| \leq \log 2),$$

we get

$$\left| e^{-u_{1,n}(y)} - e^{-u(y)} \right| \leq e^{-u(y)} \left(|v_{1,n}(y)| + v_{1,n}^2(y) \right), \quad (7)$$

if $|v_{1,n}(y)| \leq \log 2$.

Taking into account inequalities (6) and (7), from the inequality (5) we get

$$\left| F^n(a_n + b_n y) - e^{-u(y)} \right| \leq \Delta_{1,n}(y), \quad (8)$$

if $|v_{1,n}(y)| \leq \log 2$.

Analogously, we get

$$\begin{aligned} & \left| (F(a_n + b_n y) - F(c_{1,n} + d_{1,n}x_1, a_{2,n} + b_{2,n}x_2))^n - e^{-u(y)-z_1(x_1)} \right| \leq \\ & \leq \Delta_{2,n}(y, x_1), \end{aligned} \quad (9)$$

if $|v_{2,n}(y, x_1)| \leq \log 2$;

$$\begin{aligned} & \left| (F(a_n + b_n y) - F(a_{1,n} + b_{1,n}x_1, c_{2,n} + d_{2,n}x_2))^n - e^{-u(y)-z_2(x_2)} \right| \leq \\ & \leq \Delta_{3,n}(y, x_2), \end{aligned} \quad (10)$$

if $|v_{3,n}(y, x_2)| \leq \log 2$;

$$\begin{aligned} & \left| (\Delta_{[c_n + d_n x, a_n + b_n y]} F)^n - e^{-u(y)-z(x)} \right| \leq \\ & \leq \Delta_{4,n}(y, x_2), \end{aligned} \quad (11)$$

if $|v_{4,n}(y, x_2)| \leq \log 2$.

Taking into account inequalities (8) – (11), from the inequality (4) we get the estimate in (3).

Theorem is proved.

3 The Example

Let $\{X_j, j \geq 1\}$ be independent identically distributed bivariate random variables with a distribution function

$$F(x) = 1 - e^{-x_1} - e^{-x_2} + e^{-x_1 - x_2}, \quad x_1 > 0, x_2 > 0.$$

For the chosen centralizing and normalizing vectors

$$\begin{aligned} a_n &= (\log n, \log n), & b_n &= (1, 1), \\ c_n &= (0, 0), & d_n &= (1/n, 1/n), \end{aligned}$$

we have

$$\lim_{n \rightarrow \infty} P(Z_n < a_n + b_n y, W_n < c_n + d_n x) = H(y)L(x); \quad (12)$$

here

$$H(y) = \exp(-e^{-y_1} - e^{-y_2}), \quad y_1, y_2 \in \mathbf{R},$$

$$L(x) = 1 - e^{-x_1} - e^{-x_2} + e^{-x_1 - x_2}, \quad x_1 > 0, x_2 > 0.$$

We shall estimate the convergate rate in (12). We have

$$u_{1,n}(y) = e^{-y_1} + e^{-y_2} - \frac{e^{-y_1 - y_2}}{n},$$

$$v_{1,n}(y) = -\frac{e^{-y_1 - y_2}}{n},$$

$$u_{2,n}(y, x_1) = e^{-y_1} + e^{-y_2 - x_1/n} - \frac{e^{-y_1 - y_2}}{n} + n \left(1 - e^{-x_1/n} \right),$$

$$v_{2,n}(y, x_1) = e^{-y_2} \left(e^{-x_1/n} - 1 \right) - \frac{e^{-y_1 - y_2}}{n} + n \left(1 - e^{-x_1/n} \right) - x_1,$$

$$u_{3,n}(y, x_2) = e^{-y_2} + e^{-y_1 - x_2/n} - \frac{e^{-y_1 - y_2}}{n} + n \left(1 - e^{-x_2/n} \right),$$

$$v_{3,n}(y, x_2) = e^{-y_1} \left(e^{-x_2/n} - 1 \right) - \frac{e^{-y_1 - y_2}}{n} + n \left(1 - e^{-x_2/n} \right) - x_2,$$

$$u_{4,n}(y, x) = e^{-y_1 - x_2/n} + e^{-y_2 - x_1/n} - \frac{e^{-y_1 - y_2}}{n} + n \left(1 - e^{-(x_1 - x_2)/n} \right),$$

$$v_{4,n}(y, x) = e^{-y_1} \left(e^{-x_2/n} - 1 \right) + e^{-y_2} \left(e^{-x_1/n} - 1 \right) - \frac{e^{-y_1 - y_2}}{n} + n \left(1 - e^{-(x_1 + x_2)/n} \right) - x_1 - x_2.$$

Evidently, the order of the convergence rate with respect to n equals $1/n$.

4 References

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